BLOW-UP CONDITIONS FOR TWO DIMENSIONAL MODIFIED EULER-POISSON EQUATIONS

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Abstract. The multi-dimensional Euler-Poisson system describes the dynamic behavior of many important physical flows, yet as a hyperbolic system its solution can blow-up for some initial configurations. This article strives to advance our understanding on the critical threshold phenomena through the study of a two-dimensional modified Euler-Poisson system with a modified Riesz transform where the singularity at the origin is removed. We identify upper-thresholds for finite time blow-up of solutions for the modified Euler-Poisson equations with attractive/repulsive forcing.

1. Introduction

We are concerned with the threshold phenomenon in two-dimensional Euler-Poisson equations. The pressure-less Euler-Poisson (EP) equations in multi-dimensions are

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.1a}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = k \nabla \Delta^{-1} \rho, \tag{1.1b}$$

which are the usual statements of the conservation of mass and Newton's second law. Here k is a physical constant which parameterizes the repulsive $k > 0$ or attractive $k < 0$ forcing. The local density $\rho = \rho(t, \vec{x}) : \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^+$ and the velocity field $\mathbf{u}(t, \vec{x}) : \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ are the unknowns. This hyperbolic system (1.1) with nonlocal forcing describes the dynamic behavior of many important physical flows, including plasma with collision, cosmological waves, charge transport, and the collapse of stars due to self gravitation.

There is a considerable amount of literature available on the solution behavior of Euler-Poisson system. Let us mention the study of steady-state solutions [15] and the global existence of weak solutions [16]. Global existence due to damping relaxation and with non-zero background can be found in [18]. Construction of a global smooth irrotational solution in three and two dimensional space can be found in [5, 7].

To address the fundamental question of the persistence of C^1 regularity for solutions of the Euler-Poisson system and related models, the concept of Critical Threshold (CT) is originated and developed in a series of papers by Engelberg, Liu and Tadmor [4, 11, 12, 13, 14] and more. The critical threshold in [4] describes the conditional stability of the one-dimensional Euler-Poisson system, where the answer to the question of global vs. local existence depends on whether the initial data crosses a critical threshold. Following [4], critical thresholds have been identified for several one-dimensional models, including 2×2

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quasi-linear hyperbolic relaxation systems [10], Euler equations with non-local interaction and alignment forces [1], and traffic flow models [9].

Moving to the multi-dimensional setup, one needs to identify the proper quantities to describe the critical threshold phenomenon. Liu and Tadmor introduce in [11] the method of spectral dynamics which relies on the dynamical system governing eigenvalues of the velocity gradient matrix, $M := \nabla \mathbf{u}$, along particle paths. In order to trace the evolution of $M := \nabla \mathbf{u}$, we differentiate (1.1b), obtaining

$$\partial_t M + \mathbf{u} \cdot \nabla M + M^2 = k \nabla \otimes \nabla \Delta^{-1} \rho = kR[\rho], \tag{1.2}$$

where $R[\cdot]$ is the 2×2 Riesz matrix operator, defined as

$$R[h] := \nabla \otimes \nabla \Delta^{-1}[h] = \mathcal{F}^{-1} \left\{ \frac{\xi_i \xi_j}{|\xi|^2} \hat{h}(\xi) \right\}_{i,j=1,2}.$$

We let $\frac{D}{Dt}[\cdot] = [\cdot]'$ be the usual material derivative, $\frac{\partial}{\partial t} + u \cdot \nabla$. We are concerned with the initial value problem (1.2) or

$$\frac{D}{Dt} M + \begin{pmatrix} M_{11}^2 + M_{12}M_{21} & dM_{12} \\ dM_{21} & M_{12}M_{21} + M_{22}^2 \end{pmatrix} = k \begin{pmatrix} R_{11}[\rho] & R_{12}[\rho] \\ R_{21}[\rho] & R_{22}[\rho] \end{pmatrix}. \tag{1.3}$$

subject to initial data

$$(M, \rho)(0, \cdot) = (M_0, \rho_0).$$

The global nature of the Riesz matrix $R[\rho]$, makes the issue of regularity for Euler-Poisson equations such an intricate question to solve.

This work propose a modified Euler-Poisson system as an effort to gain a better understanding on Euler-Poisson equation (1.3). Before we introduce the modified Euler-Poisson system, we introduce several quantities with which we characterize the behavior of the velocity gradient tensor M. These are the trace $d := \mathrm{tr} M = \nabla \cdot \mathbf{u}$, the vorticity $\omega := \nabla \times \mathbf{u} = M_{21} - M_{12}$ and nonlinear quantities $\eta := M_{11} - M_{22}$ and $\xi := M_{12} + M_{21}$.

Taking the trace of (1.3), one obtain

$$\begin{aligned} d' &= -(M_{11}^2 + M_{22}^2) - 2M_{12}M_{21} + k(R_{11}[\rho] + R_{22}[\rho]) \\ &= -\left\{ \frac{(M_{11} + M_{22})^2}{2} + \frac{(M_{11} - M_{22})^2}{2} \right\} + \frac{(M_{21} - M_{12})^2}{2} - \frac{(M_{12} + M_{21})^2}{2} + k\rho \\ &= -\frac{1}{2} d^2 - \frac{1}{2} \eta^2 + \frac{1}{2} \omega^2 - \frac{1}{2} \xi^2 + k\rho. \end{aligned} \tag{1.4}$$

From the matrix equation (1.3), and (1.1a), we obtain

$$\eta' + \eta d = k(R_{11}[\rho] - R_{22}[\rho]), \tag{1.5a}$$

$$\omega' + \omega d = k(R_{21}[\rho] - R_{12}[\rho]) = 0, \tag{1.5b}$$

$$\xi' + \xi d = k(R_{12}[\rho] + R_{21}[\rho]), \tag{1.5c}$$

$$\rho' + \rho d = 0. \tag{1.5d}$$

From (1.5b) and (1.6a), we derive

$$\frac{\omega}{\omega_0} = \frac{\rho}{\rho_0}.$$

This allows us to rewrite the system (1.4) and (1.1a), i.e.

$$d' = -\frac{1}{2}d^2 + \frac{1}{2}\Big(\frac{\omega_0}{\rho_0}\Big)^2\rho^2 - \frac{1}{2}\eta^2 - \frac{1}{2}\xi^2 + k\rho, \tag{1.6a}$$

$$\rho' = -\rho d \tag{1.6b}$$

We can see that the equation (1.4) is the Ricatti-type equation; one can view the dynamics of d as the result of a contest between negative and positive terms in (1.4). For example, one might think bigger $|\omega|$(correspond to the size of vorticity) prevents the finite time blow-up as opposed to the bigger η, ξ help the finite time blow-up.

To put our study in a proper perspective we recall a few recent works in the form of (1.6). Once we re-write the ω^2 term using ρ as written in (1.6a) one can see that by setting $\omega_0 = 0$ i.e., assuming *vanishing initial vorticity*, and dropping $-\eta^2$, $-\xi^2$ terms, (1.6a) is reduced to simple Ricatti-type inequality

$$d' \leq -\frac{d^2}{2} + k\rho.$$

Using this argument, Chae and Tadmor [2] proved the finite time blow-up for solutions of $k < 0$ case in arbitrary space dimension. Later Cheng and Tadmor [3] improved the result of [2] using the delicate ODE phase plane argument.

Turning to the non-vanishing initial vorticity case, one need to investigate the competition between ρ^2, η^2 and ξ^2 terms. Apparently the two latter terms help the blow-up of d. However, we have no clear idea on how fast η and ξ terms are changing in time. This is because (see (1.5a) and (1.5c)) we are lack of L^∞ bound of $R_{ij}[\rho]$. To gain better understanding of the dynamics of d, Liu and Tadmor introduce the restricted Euler-Poisson(REP) system[11, 12], which is obtained from (1.3) by restricting attention to the local isotropic trace $\frac{k}{2}\rho I_{2\times 2}$ of the global coupling term $kR[\rho]$. Then in the REP system, (1.5a) and (1.5c) are changed to

$$\eta' = -\eta d, \ and \ \xi' = -\xi d,$$

respectively. Thus (1.6a) is reduced to

$$d' = -\frac{1}{2}d^2 + \frac{1}{2}\Big(\frac{\omega_0}{\rho_0}\Big)^2\rho^2 - \frac{1}{2}\Big(\frac{\eta_0}{\rho_0}\Big)^2\rho^2 - \frac{1}{2}\Big(\frac{\xi_0}{\rho_0}\Big)^2\rho^2 + k\rho,$$

or

$$d' + \frac{d^2 + \beta\rho^2}{2} = k\rho.$$

They studied the dynamics of (ρ, d) parametrized by β, and it was shown that in the repulsive case, the restricted two-dimensional REP system admits two-sided critical threshold. For arbitrary $n \geq 3$ dimensional REP system, the author and Liu identified both upper-thresholds for finite time blow-up of solutions and sub-thresholds for global existence of solutions [8].

In this work, we propose a two-dimensional *modified* Euler-Poission(MEP) system with a modified Riesz transform where singularity at origin is removed. We identify upper-thresholds for finite time blow-up of solutions for the MEP system with attractive and repulsive forcing.

As noted earlier, the main obstacle in handling (1.3) is the lack of an accurate description for the propagation of the Riesz transform. The modified Riesz transform in the

MEP system is intended to take into account the *global* forcing in the full Euler-Poisson equations, as opposed to the REP systems in [8, 11, 12] are *localized* Euler-Poisson equations.

In [8, 11, 12], multi-dimensional REP system's blow-up conditions depend on the relatives sizes of the following quantities: the initial density, the initial divergence and the eigenvalues of the initial velocity gradient matrix. In addition to these initial quantities, for the blow-up of the MEP system, the relative size of the *initial total mass* plays an important role.

The rest of this paper is organized as follows. In section 2, we introduce a Modified Riesz transform and study the Euler-Poisson equations with the modified Riesz transform. We state our main results about finite time blow-up of solutions to the modified Euler-Poission system. The details of the proofs of those main results are carried out in Section 3. Finally in the appendix, we discuss the detailed calculation of the Riesz transform.

2. Modified Riesz transform and statement of main results

In this section we start with introducing the Modified Riesz transform and discuss some motivation behind the definition.

One can explicitly calculate $\frac{\partial^2}{\partial x_j \partial x_i}\Delta^{-1}h(\vec{x})$, i.e.,

$$(R_{ij}[h])(\vec{x}) = p.v. \int_{\mathbb{R}^2} \frac{\partial^2}{\partial y_j \partial y_i} G(\vec{y}) h(\vec{x}-\vec{y})\, d\vec{y} + \frac{h(\vec{x})}{2\pi} \int_{|\vec{z}|=1} z_i z_j \, d\vec{z}, \tag{2.1}$$

where $G(\vec{y})$ is the Poisson kernel in two-dimensions, and is given by

$$G(\vec{y}) = \frac{1}{2\pi} \log|\vec{y}|.$$

The detailed calculations are given in the Appendix. Due to the singular nature of the integral, we are lack of L^∞ estimate of the $R_{ij}[h]$.

We try to approximate R_{ij} by replacing the Cauchy principle integral by the integral over the $\mathbb{R}^2 \setminus B(0,\nu)$(i.e., an open origin-centered disk of pre-determined radius ν is removed from $\mathbb{R}^2$). That is, we define the modified Riesz transform as follows:

Definition 2.1. (The modified Riesz transform) Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with $\|h\|_{L^1(\mathbb{R}^2)} < \infty$. For $0 < \nu << 1$, we define the modified Riesz transform as follow:

$$(R^\nu_{ij}[h])(\vec{x}) := \int_{\mathbb{R}^2 \setminus B(0,\nu)} \frac{\partial^2}{\partial y_i \partial y_j} G(\vec{y}) h(\vec{x}-\vec{y})\, d\vec{y} + \frac{h(\vec{x})}{2\pi} \int_{|\vec{z}|=1} z_i z_j \, d\vec{z},$$

where $G(\vec{x}) = \frac{1}{2\pi}\log|\vec{x}|$ is the Green's function for the Poisson equation in two-dimensions.

Remark) Some remarks are in order at this point.

i) For our blow-up analysis one can let ν be very small, as long as ν is fixed. One may obtain $R^\nu_{ij}[h] \to R_{ij}[h]$ as $\nu \to 0$ for a smooth function h. So one may consider $R^\nu_{ij}[h]$ as an approximation of $R_{ij}[h]$.

ii) As we recover ρ from the trace of the right hand side of (1.3), i.e., $R_{11}[\rho]+R_{22}[\rho] = \rho$, we obtain the same result with the modified Riesz transform,

$$R^\nu_{11}[h] + R^\nu_{22}[h] = h.$$

iii) Since we removed the singular integral issue by restricting attention to $\mathbb{R}^2 \setminus B(\vec{0}, \nu)$, we will later estimate the integral using the L^1 norm of h.

From now on, we are concerned with the initial value problem of the modified Euler-Poisson(MEP) system

$$\frac{D}{Dt}M + \begin{pmatrix} M_{11}^2 + M_{12}M_{21} & dM_{12} \\ dM_{21} & M_{12}M_{21} + M_{22}^2 \end{pmatrix} = k \begin{pmatrix} R_{11}^{\nu}[\rho] & R_{12}^{\nu}[\rho] \\ R_{21}^{\nu}[\rho] & R_{22}^{\nu}[\rho] \end{pmatrix}, \tag{2.2a}$$

$$\frac{D}{Dt}\rho + \rho \mathrm{tr} M = 0, \tag{2.2b}$$

subject to initial data

$$(M, \rho)(0, \cdot) = (M_0, \rho_0).$$

Our goal of this work is to prove the following results.

Theorem 2.2. *Consider the two-dimensional attractive MEP system* (2.2) *with* $k < 0$. *Suppose that* $\rho(0, \cdot) \in L^1(\mathbb{R}^2)$, $d_0 < 0$ *and* $\rho_0 > 0$. *If there exist a constant* μ *such that*

$$\frac{|\omega_0|}{\rho_0} < \mu < \frac{\sqrt{\eta_0^2 + \xi_0^2}}{\rho_0},$$

and

$$F(\mu, d_0, \omega_0, \rho_0, \eta_0, \xi_0, \|\rho(0, \cdot)\|_{L^1(\mathbb{R}^2)}) \geq 0,$$

then $d(t)$ *and* $\rho(t)$ *must blow-up at some finite time. Here* F *is given by,*

$$\begin{aligned} F(\mu, d, \omega, \rho, \eta, \xi, \|\rho(0, \cdot)\|_{L^1(\mathbb{R}^2)}) := & \frac{\pi\nu^2}{\sqrt{2}|k|\|\rho(0, \cdot)\|_{L^1(\mathbb{R}^2)}} \left(\sqrt{\eta^2 + \xi^2} - \rho\mu\right) \\ & - \frac{\pi + 2\arctan(d/\sqrt{\mu^2\rho^2 - \omega^2 - 2k\rho})}{\sqrt{\mu^2\rho^2 - \omega^2 - 2k\rho}}. \end{aligned}$$

Theorem 2.3. *Consider the two-dimensional repulsive MEP system* (2.2) *with* $k > 0$. *Suppose that* $\rho(0, \cdot) \in L^1(\mathbb{R}^2)$, $d_0 < 0$ *and* $\rho_0 > 0$. *If there exist a constant* μ *such that*

$$\sqrt{\left(\frac{\omega_0}{\rho_0}\right)^2 + \frac{2k}{\rho_0}} < \mu < \frac{\sqrt{\eta_0^2 + \xi_0^2}}{\rho_0},$$

and

$$F(\mu, d_0, \omega_0, \rho_0, \eta_0, \xi_0, \|\rho(0, \cdot)\|_{L^1(\mathbb{R}^2)}) \geq 0,$$

then $d(t)$ *and* $\rho(t)$ *must blow-up at some finite time. Here* F *is given in Theorem 2.2.*

Remark) Some remarks are in order at this point.

(i) We first notice that the set of initial configurations that satisfy $F \geq 0$ is a *non-empty* set. Indeed, the first term

$$\frac{\pi\nu^2}{\sqrt{2}|k|\|\rho(0, \cdot)\|_{L^1(\mathbb{R}^2)}} \left(\sqrt{\eta^2 + \xi^2} - \rho\mu\right)$$

of F is positive, and the second term

$$-\frac{\pi + 2\arctan(d/\sqrt{\mu^2\rho^2 - \omega^2 - 2k\rho})}{\sqrt{\mu^2\rho^2 - \omega^2 - 2k\rho}}$$

is negative because $|\arctan(\cdot)| < \frac{\pi}{2}$. The second term approaches zero as d goes $-\infty$, therefore, for sufficiently small d the condition $F \geq 0$ is ensured.

(ii) The critical threshold in one-dimensional Euler-Poisson equations[4] depends only on the relative size of the initial velocity gradient and initial density. In contrast to the one-dimensional Euler-Poisson equations, the threshold conditions in two-dimensional MEP equations depend on several initial quantities: density ρ_0, divergence d_0, vorticity ω_0, gaps η_0, ξ_0 and even total mass $\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}$.

(iii) One can easily check that how F depends on those initial configurations:

$$\frac{\partial F}{\partial d} < 0, \ \frac{\partial F}{\partial(\omega^2)} < 0, \ \frac{\partial F}{\partial \rho} > 0, \ \frac{\partial F}{\partial \|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}} > 0, \ \frac{\partial F}{\partial \eta} > 0, \ \textit{and} \ \frac{\partial F}{\partial \xi} > 0.$$

For example, F is increasing in ρ, $\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}$ and $-d$. This is interpreted as if there is a point $\vec{x} \in \mathbb{R}^2$ with highly accumulated mass with low divergence, then there may be a finite time blow-up of the density.

(iv) The condition $\frac{\partial F}{\partial(\omega^2)} < 0$ is understood as to ensure the finite time blow-up, relatively small size of initial vorticity $|\nabla \times \mathbf{u}_0|$ is needed. This fact is consistent with the results in so-called restricted type flows(e.g. [12], [11] and [13]); especially the result in [12] show that the global smooth solution is ensured if the initial velocity gradient has complex eigenvalues, which applies, for example, for a class of initial configurations with sufficiently large vorticity.

(v) By setting ν small, one may use Theorems 2.2 and 2.3 to understand the blow-up phenomenon for the *full* Euler-Poisson equations. But as $\nu \searrow 0$ one can see that the first term

$$\frac{\pi\nu^2}{\sqrt{2}|k|\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}}\big(\sqrt{\eta^2+\xi^2} - \rho\mu\big),$$

which is positive, of F approaches 0. So as $\nu \searrow 0$, the initial configurations set that ensure finite time blow-up is impoverished.

3. Proofs of main Theorems

We first start with solving the ODEs (1.5a) and (1.5c). We let

$$f(t) := k(R^{\nu}_{11}[\rho] - R^{\nu}_{22}[\rho]), \tag{3.1}$$

and consider (1.5a) or

$$\begin{aligned} &\eta' + \eta d = f(t) \\ &\Rightarrow \left\{ e^{\int_0^t d(s)\,ds}\eta \right\}' = f(t)e^{\int_0^t d(s)\,ds} \\ &\Rightarrow e^{\int_0^t d(s)\,ds}\eta(t) = \int_0^t f(r)e^{\int_0^\tau d(s)\,ds}\,d\tau + \eta(0). \end{aligned} \tag{3.2}$$

Thus, since

$$\rho(t) = \rho_0 e^{-\int_0^t d(s)\,ds},$$

we obtain

$$
\begin{aligned}
\eta(t) &= \eta_0 e^{-\int_0^t d(s)\,ds} + e^{-\int_0^t d(s)\,ds}\int_0^t f(r) e^{\int_0^\tau d(s)\,ds}\,d\tau \\
&= \eta_0 \frac{\rho}{\rho_0} + \frac{\rho}{\rho_0}\int_0^t f(\tau)\frac{\rho_0}{\rho}\,d\tau \\
&= \left(\frac{\eta_0}{\rho_0} + \int_0^t \frac{f(\tau)}{\rho(r,\vec{x}(\tau))}\,d\tau\right)\rho(t,\vec{x}(t)).
\end{aligned} \tag{3.3}
$$

In the sequel, we use the simple notation for $\rho(t,\vec{x}(t))$. That is

$$
\eta(t) = \left(\frac{\eta_0}{\rho_0} + \int_0^t \frac{f(\tau)}{\rho(\tau)}\,d\tau\right)\rho(t). \tag{3.4}
$$

Similarly we let

$$
g(t) := k(R^\nu_{12}[\rho] + R^\nu_{21}[\rho]) = 2kR^\nu_{12}[\rho],
$$

and solve ODE (1.5c), to obatin

$$
\xi(t) = \left(\frac{\xi_0}{\rho_0} + \int_0^t \frac{g(\tau)}{\rho(\tau)}\,d\tau\right)\rho(t). \tag{3.5}
$$

Now with (3.4) and (3.5), we rewrite the system (1.6) as follows

$$
d' = -\frac{1}{2}d^2 + \frac{1}{2}\left[\left(\frac{\omega_0}{\rho_0}\right)^2 - \left(\frac{\eta_0}{\rho_0} + \int_0^t \frac{f(\tau)}{\rho(\tau)}\,d\tau\right)^2 - \left(\frac{\xi_0}{\rho_0} + \int_0^t \frac{g(\tau)}{\rho(\tau)}\,d\tau\right)^2\right]\rho^2 + k\rho, \tag{3.6a}
$$

$$
\rho' = -\rho d. \tag{3.6b}
$$

We estimate the integrals in (3.6a):

Lemma 3.1. *For $t > 0$, it holds*

$$
\left|\int_0^t \frac{f(\tau)}{\rho(\tau)}\,d\tau\right| \le \frac{|k|\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}}{\pi\nu^2}\cdot\int_0^t \frac{1}{\rho(\tau)}\,d\tau,
$$

and

$$
\left|\int_0^t \frac{g(\tau)}{\rho(\tau)}\,d\tau\right| \le \frac{|k|\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}}{\pi\nu^2}\cdot\int_0^t \frac{1}{\rho(\tau)}\,d\tau.
$$

Proof. We first estimate $|f(t)|$ and $|g(t)|$. We note that

$$
\frac{\partial^2 G(\vec{y})}{\partial y_1^2} = \frac{1}{2\pi}\cdot\frac{-y_1^2+y_2^2}{(y_1^2+y_2^2)^2} \quad \textit{and} \quad \frac{\partial^2 G(\vec{y})}{\partial y_1^2} = \frac{1}{2\pi}\cdot\frac{y_1^2-y_2^2}{(y_1^2+y_2^2)^2}.
$$

From (3.1) we obtain,

$$
\begin{aligned}
|f(t)| &= \left|\frac{k}{\pi}\int_{\mathbb{R}^2\setminus B(0,\nu)} \frac{-y_1^2+y_2^2}{(y_1^2+y_2^2)^2}\rho(t,\vec{x}(t)-\vec{y})\,d\vec{y} + \frac{k\rho(t,\vec{x}(t))}{2\pi}\int_{|\vec{z}|=1} z_1^2 - z_2^2\,d\vec{z}\right| \\
&\le \frac{|k|}{\pi}\frac{1}{\nu^2}\|\rho(t,\cdot)\|_{L^1(\mathbb{R}^2)} + 0,
\end{aligned} \tag{3.7}
$$

here the second integral vanishes due to symmetry. Since $\|\rho(t,\cdot)\|_{L^1(\mathbb{R}^2)} = \|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}$, $\forall t \ge 0$ from (1.1a), we obtain

$$
|f(t)| \le \frac{|k|}{\pi\nu^2}\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}.
$$

For the bounds of $g(t)$, we note that $\frac{\partial^2 G(\vec{y})}{\partial y_1 \partial y_2} = \frac{1}{\pi}\frac{-y_1 y_2}{(y_1^2+y_2^2)^2}$, and the bound is obtained similarly, i.e.,

$$|g(t)| \leq \frac{|k|}{\pi\nu^2}\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}.$$

We next apply these estimates to obtain,

$$\begin{aligned} \left|\int_0^t \frac{f(\tau)}{\rho(\tau)}\,d\tau\right| &\leq \int_0^t \frac{|f(\tau)|}{\rho(\tau)}\,d\tau \\ &\leq \frac{|k|}{\pi\nu^2}\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)} \cdot \int_0^t \frac{1}{\rho(\tau)}\,d\tau, \end{aligned} \tag{3.8}$$

and thus the desired results follows. The bound of $\left|\int_0^t \frac{g(\tau)}{\rho(\tau)}\,d\tau\right|$ is obtained similarly. $\square$

Once we consider the coefficient of ρ^2 in (3.6a), for a short time interval, the coefficient is dominated by

$$\frac{1}{2}\left[\left(\frac{\omega_0}{\rho_0}\right)^2 - \left(\frac{\eta_0}{\rho_0}\right)^2 - \left(\frac{\xi_0}{\rho_0}\right)^2\right],$$

because two integrals in (3.6a) have value zero at $t = 0$. We state this observation as our key lemma:

Lemma 3.2. *For any $\mu \in (0, \frac{1}{\rho_0}\sqrt{\eta_0^2+\xi_0^2}]$, there exists $T > 0$ such that*

$$\begin{aligned} d' &\leq -\frac{1}{2}d^2 + \frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2 - \mu^2\right\}\rho^2 + k\rho, \\ \rho' &= -d\rho, \end{aligned} \tag{3.9}$$

for all $t \in [0, T]$. Furthermore, the lower bound $t^* > 0$ of T is obtained from

$$\sqrt{\left(\frac{\eta_0}{\rho_0}\right)^2 + \left(\frac{\xi_0}{\rho_0}\right)^2} - \mu = \frac{\sqrt{2}|k|\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}}{\pi\nu^2}\int_0^{t^*}\frac{1}{\rho(\tau)}\,d\tau. \tag{3.10}$$

Proof. From (3.6a), it suffices to show that

$$\left(\frac{\eta_0}{\rho_0} + \int_0^t \frac{f(\tau)}{\rho(\tau)}\,d\tau\right)^2 + \left(\frac{\xi_0}{\rho_0} + \int_0^t \frac{g(\tau)}{\rho(\tau)}\,d\tau\right)^2 \geq \mu^2, \tag{3.11}$$

for all $t \in [0, t^*]$. The left hand side is interpreted as the square of the distance between two points

$$\left(\frac{\eta_0}{\rho_0}, \frac{\xi_0}{\rho_0}\right) \quad \textit{and} \quad \left(-\int_0^t \frac{f(\tau)}{\rho(\tau)}\,d\tau, -\int_0^t \frac{g(\tau)}{\rho(\tau)}\,d\tau\right)$$

on $\mathbb{R}^2$.

For any time t, the latter point is located within the origin-centered disk of radius

$$\frac{\sqrt{2}|k|\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}}{\pi\nu^2} \cdot \int_0^t \frac{1}{\rho(\tau)}\,d\tau,$$

due to Lemma 3.1. We note that the disk's radius is 0 when $t = 0$, and the disk is expanding in time. The lower bound t^* is the instance when the expanding disk intersects with $(\frac{\eta_0}{\rho_0}, \frac{\xi_0}{\rho_0})$-centered disk of radius μ. Therefore, we obtain (3.10), and for any $t \in [0, t^*]$, the inequality (3.11) holds. This completes the proof. $\square$

Proof of Theorem 2.2 : We suppose that $k < 0$, $d_0 < 0$ and $\rho_0 > 0$. It is clear to have that if

$$\left(\frac{\omega_0}{\rho_0}\right)^2 - \mu^2 < 0,$$

then from (3.9), $d(t) < 0$ for all $t \leq [0, t^*]$. Thus by the second equation in (3.9), $\rho(t)$ is strictly increasing in t and from (3.10), it follows that

$$\sqrt{\left(\frac{\eta_0}{\rho_0}\right)^2 + \left(\frac{\xi_0}{\rho_0}\right)^2} - \mu \leq \frac{\sqrt{2}|k|\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}}{\pi\nu^2}\frac{1}{\rho_0} \cdot t^*.$$

This inequality allow us to obtain the explicit lower bound for t^*, i.e.,

$$t^* \geq T^* := \frac{\pi\nu^2\rho_0}{\sqrt{2}|k|\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}}\left\{\sqrt{\left(\frac{\eta_0}{\rho_0}\right)^2 + \left(\frac{\xi_0}{\rho_0}\right)^2} - \mu\right\}. \tag{3.12}$$

Now, by Lemma 3.2, it holds

$$\begin{aligned} d' &\leq -\frac{1}{2}d^2 + \frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2 - \mu^2\right\}\rho^2 + k\rho \\ &\leq -\frac{1}{2}d^2 + \frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2 - \mu^2\right\}\rho_0^2 + k\rho_0, \end{aligned} \tag{3.13}$$

for all $t \in [0, T^*]$. Here the last inequality holds because $\rho(t)$ is strictly increasing and both k and $\left(\frac{\omega_0}{\rho_0}\right)^2 - \mu^2$ are negative. One can easily see that $d_0 < 0$ leads to finite time blow-up of $d(t)$. But we note that the finite time blow-up must occurs before T^*. We therefore let

$$N := \frac{1}{2}\left\{\mu^2 - \left(\frac{\omega_0}{\rho_0}\right)^2\right\}\rho_0^2 - k\rho_0,$$

(N is non-negative) and explicitly solve the ordinary differential inequality

$$d' \leq -\frac{1}{2}d^2 - N \tag{3.14}$$

to obtain

$$d(t) \leq -\sqrt{2N}\tan\left(\sqrt{\frac{N}{2}}t - \arctan\left(\frac{d_0}{\sqrt{2N}}\right)\right). \tag{3.15}$$

Since $N > 0$, it follows that if $d_0 < 0$, then $d(t) \to -\infty$ as

$$t \to \frac{\frac{\pi}{2} + \arctan(\frac{d_0}{\sqrt{2N}})}{\sqrt{N/2}}.$$

By requiring $\frac{\frac{\pi}{2}+\arctan(\frac{d_0}{\sqrt{2N}})}{\sqrt{N/2}} \leq T^*$, we obtain the blow-up condition in the Theorem 2.2.

Proof of Theorem 2.3 : We suppose that $k > 0$, $d_0 < 0$ and

$$\rho_0 \geq \rho^* := \frac{2k}{\mu^2 - (\omega_0/\rho_0)^2}. \tag{3.16}$$

These implies

$$\left(\frac{\omega_0}{\rho_0}\right)^2 - \mu^2 \le -\frac{2k}{\rho_0} < 0.$$

Therefore,

$$\frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2 - \mu^2\right\}\rho^2 + k\rho \le 0,$$

for all $\rho \ge \rho^*$. Thus from Lemma 3.2, $d(t) < 0$ for all $t \in [0, t^*]$ and $\rho(t)$ is strictly increasing in t. Therefore, (3.10) gives

$$\sqrt{\left(\frac{\eta_0}{\rho_0}\right)^2 + \left(\frac{\xi_0}{\rho_0}\right)^2} - \mu \le \frac{\sqrt{2}|k|\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}}{\pi\nu^2}\frac{1}{\rho_0} \cdot t^*.$$

This inequality allow us to obtain the explicit lower bound for t^*, i.e.,

$$t^* \ge T^* := \frac{\pi\nu^2\rho_0}{\sqrt{2}|k|\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}}\left\{\sqrt{\left(\frac{\eta_0}{\rho_0}\right)^2 + \left(\frac{\xi_0}{\rho_0}\right)^2} - \mu\right\}. \tag{3.17}$$

We notice that T^* is equivalent to that in the proof of Theorem 2.2. Also, with the choice of ρ_0 satisfies (3.16), the same ordinary differential inequality in (3.14) holds. Therefore we obtain the desired result.

4. Appendix - Derivation of $R_{ij}[\cdot]$

In this appendix section we calculate $R_{ij}[\cdot]$, i.e, the 2nd order derivatives of $\Delta^{-1}h(\vec{x})$. A similar calculation can be found in [17].

We first consider the 1st order derivative:

$$\begin{aligned}
\frac{\partial}{\partial x_i}\Delta^{-1}h(\vec{x}) &= p.v.\int_{\mathbb{R}^2} G(\vec{y})\frac{\partial}{\partial x_i}h(\vec{x}-\vec{y})\,d\vec{y} \\
&= \lim_{\epsilon\to 0}\left[\int_{|\vec{y}|\ge\epsilon} G(\vec{y})\frac{\partial}{\partial x_i}h(\vec{x}-\vec{y})\,d\vec{y}\right] \\
&= \lim_{\epsilon\to 0}\left[\int_{|\vec{y}|\ge\epsilon} -G(\vec{y})\frac{\partial}{\partial y_i}h(\vec{x}-\vec{y})\,d\vec{y}\right] \\
&= \lim_{\epsilon\to 0}\left[-\int_{|\vec{y}|=\epsilon} G(\vec{y})h(\vec{x}-\vec{y})\frac{-y_i}{\epsilon}\,d\vec{y} + \int_{|\vec{y}|\ge\epsilon}\frac{\partial}{\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\,d\vec{y}\right] \\
&= \lim_{\epsilon\to 0}\int_{|\vec{y}|=\epsilon} G(\vec{y})h(\vec{x}-\vec{y})\frac{y_i}{\epsilon}\,d\vec{y} + p.v.\int_{\mathbb{R}^2}\frac{\partial}{\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\,d\vec{y} \\
&= p.v.\int_{\mathbb{R}^2}\frac{\partial}{\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\,d\vec{y},
\end{aligned} \tag{4.1}$$

where the last equality holds because

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{|\vec{y}|=\epsilon} G(\vec{y})h(\vec{x}-\vec{y})\frac{y_i}{\epsilon}\,d\vec{y} &= \lim_{\epsilon\to 0}\int_{|\vec{z}|=1} G(\epsilon\vec{z})h(\vec{x}-\epsilon\vec{z})z_i\epsilon\,d\vec{z} \\
&= h(\vec{x})\lim_{\epsilon\to 0}\int_{|\vec{z}|=1}\epsilon G(\epsilon\vec{z})z_i\,d\vec{z} = 0.
\end{aligned} \tag{4.2}$$

We then calculate the 2nd order derivative:
(4.3)
$$\begin{aligned}
\frac{\partial^2}{\partial x_j \partial x_i}\Delta^{-1}h(\vec{x}) &= \frac{\partial}{\partial x_j}\left[p.v.\int_{\mathbb{R}^2}\frac{\partial}{\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\,d\vec{y}\right]\\
&= \lim_{\epsilon\to 0}\left[\int_{|\vec{y}|\geq\epsilon}\frac{\partial}{\partial y_i}G(\vec{y})\frac{\partial}{\partial x_j}h(\vec{x}-\vec{y})\,d\vec{y}\right]\\
&= \lim_{\epsilon\to 0}\left[\int_{|\vec{y}|\geq\epsilon}-\frac{\partial}{\partial y_i}G(\vec{y})\frac{\partial}{\partial y_j}h(\vec{x}-\vec{y})\,d\vec{y}\right]\\
&= \lim_{\epsilon\to 0}\left[-\int_{|\vec{y}|=\epsilon}\frac{\partial}{\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\frac{-y_j}{\epsilon}\,d\vec{y}+\int_{|\vec{y}|\geq\epsilon}\frac{\partial^2}{\partial y_j\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\,d\vec{y}\right]\\
&= \lim_{\epsilon\to 0}\int_{|\vec{y}|=\epsilon}\frac{\partial}{\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\frac{y_j}{\epsilon}\,d\vec{y}+p.v.\int_{\mathbb{R}^2}\frac{\partial^2}{\partial y_j\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\,d\vec{y}\\
&= \lim_{\epsilon\to 0}\int_{|\vec{z}|=1}\frac{\partial}{\partial y_i}G(\epsilon\vec{z})h(\vec{x}-\epsilon\vec{z})z_j\epsilon\,d\vec{z}+p.v.\int_{\mathbb{R}^2}\frac{\partial^2}{\partial y_j\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\,d\vec{y}\\
&= h(\vec{x})\lim_{\epsilon\to 0}\int_{|\vec{z}|=1}\frac{1}{\epsilon}\frac{\partial}{\partial z_i}G(\epsilon\vec{z})z_j\epsilon\,d\vec{z}+p.v.\int_{\mathbb{R}^2}\frac{\partial^2}{\partial y_j\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\,d\vec{y}\\
&= h(\vec{x})\lim_{\epsilon\to 0}\int_{|\vec{z}|=1}\frac{\partial}{\partial z_i}G(\epsilon\vec{z})z_j\,d\vec{z}+p.v.\int_{\mathbb{R}^2}\frac{\partial^2}{\partial y_j\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\,d\vec{y}\\
&= h(\vec{x})\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{|\vec{z}|=1}\frac{z_i}{z_1^2+z_2^2}\cdot z_j\,d\vec{z}+p.v.\int_{\mathbb{R}^2}\frac{\partial^2}{\partial y_j\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\,d\vec{y}.
\end{aligned}$$

Therefore,

$$\frac{\partial^2}{\partial x_j\partial x_i}\Delta^{-1}h(\vec{x}) = h(\vec{x})\frac{1}{2\pi}\int_{|\vec{z}|=1}z_iz_j\,d\vec{z}+p.v.\int_{\mathbb{R}^2}\frac{\partial^2}{\partial y_j\partial y_i}G(\vec{y})h(\vec{x}-\vec{y})\,d\vec{y}. \tag{4.4}$$

Since

$$\frac{\partial^2}{\partial z_1^2}G(\vec{z}) = \frac{1}{2\pi}\cdot\frac{-z_1^2+z_2^2}{(z_1^2+z_2^2)^2}\ \textit{and}\ \frac{\partial^2}{\partial z_2^2}G(\vec{z}) = \frac{1}{2\pi}\cdot\frac{-z_2^2+z_1^2}{(z_1^2+z_2^2)^2},$$

from (4.4), we notice that

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\Delta^{-1}h(\vec{x}) = h(\vec{x})+0 = h(\vec{x}). \tag{4.5}$$

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